ON SOME GENERAL SOLUTIONS OF THE SIMPLE PELL EQUATION

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Abstract. Two theorems are demonstrated giving analytical expressions of the fundamental solutions of the Pell equation $X^2 - DY^2 = 1$ found by the method of continued fractions for two squarefree polynomial expressions of radicands of Richaud-Degert type D of the form $D = (f(u))^2 \pm 2^\alpha n$, where D, $n > 0$, $\alpha \geq 0, \in \mathbb{Z}$, and $f(u) > 0, \in \mathbb{Z}$, any polynomial function of $u \in \mathbb{Z}$ such that $f(u) \equiv 0 \left(mod\ (2^{\alpha-1} n)\right)$ or $f(u) \equiv \left(2^{\alpha-2} n\right) \left(mod\ \left(2^{\alpha-1} n\right)\right)$.

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1. Introduction

Pell equations of the general form

$$X^2 - DY^2 = N \tag{1.1}$$

with $X, Y, N \in \mathbb{Z}$ and squarefree $D > 0, \in \mathbb{Z}$, have been investigated in various forms since long, already by Indian and Greek mathematicians and further in the 17th and 18th centuries by Pell, Brouckner, Lagrange, Euler and others (see historical accounts in [1], [6][1], [7], [14]) and are treated in several classical text books (see e.g. [3], [9], [10], [15] and references therein).
For $N = 1$, the simple Pell equation reads classically

$$X^2 - DY^2 = 1 \tag{1.2}$$

Beside the obvious trivial solution $(X_t, Y_t) = (1, 0)$, a whole infinite branch of solutions exists for $n > 0, \in \mathbb{Z}$ given by

$$\begin{aligned} X_n &= \frac{\left(X_1 + \sqrt{D}Y_1\right)^n + \left(X_1 - \sqrt{D}Y_1\right)^n}{2} \\ Y_n &= \frac{\left(X_1 + \sqrt{D}Y_1\right)^n - \left(X_1 - \sqrt{D}Y_1\right)^n}{2\sqrt{D}} \end{aligned} \tag{1.3}$$

where (X_1, Y_1) is the fundamental solution to (1.2), i.e. the smallest integer solution $(X_1 > 1, Y_1 > 0, \in \mathbb{Z}^+)$ different from the trivial solution. Robertson [12] lists five methods to find the fundamental solution (X_1, Y_1). Among these, the classical method introduced by Lagrange [5], based on the continued fraction expansion of the quadratic irrational $\sqrt{D}$, is at the heart of several other methods (see also [8]) . In this paper, we demonstrate two theorems yielding analytical expressions of the fundamental solutions (X_1, Y_1) of the simple Pell equation for squarefree polynomial expressions of radicands of Richaud-Degert type D of the form $D = (f(u))^2 \pm 2^\alpha n$

[1][6] lists 138 references of articles on Pell equations from 1658 to 1943.

where $D, n \in \mathbb{Z}^+$, $\alpha \geq 0 \in \mathbb{Z}^*$ and $f(u)$ any polynomial function of $u \in \mathbb{Z}$ with $f(u) > 0, \in \mathbb{Z}^+$ such that first, $f(u) \equiv 0 \left(mod\ \left(2^{\alpha-1}n\right)\right)$ and second, $f(u) \equiv \left(2^{\alpha-2}n\right) \left(mod\ \left(2^{\alpha-1}n\right)\right)$.

2. Solutions of the Pell equation by the continued fraction method

Quadratic irrationals x of the form

$$x = \frac{\sqrt{D}+b}{c} \tag{2.1}$$

where b, $c \neq 0$ and $D > 0, \in \mathbb{Z}$, $x \in \mathbb{R}$, and D is not a perfect square ($\sqrt{D} \notin \mathbb{Z}$), have been studied extensively (see e.g. [2], [4], [11], [13]). It is known also that a real number x has an eventually periodic regular continued fraction expansion if and only if x is a quadratic irrational. It yields that for every squarefree D, there exists $r > 0, \in \mathbb{Z}^+$ such that the regular continued fraction expansion of $\sqrt{D}$ is given by

$$\sqrt{D} = [a_0; \overline{a_1, \ldots, a_r, 2a_0}] \tag{2.2}$$

where $a_0 = \left\lfloor \sqrt{D} \right\rfloor$, the greatest integer $\leq \sqrt{D}$; the a_i ($1 \leq i \leq r$, where i and $a_i > 0, \in \mathbb{Z}^+$) are the partial denominators of the continued fraction expansion; the finite sequence $a_1, \ldots, a_r$ is symmetric (so $a_j = a_{r-j+1}$ for $1 \leq j \leq \lfloor r/2 \rfloor$, with $j > 0, \in \mathbb{Z}^+$); and the bar indicates the period of length $r+1$. We use then the algorithm given by Sierpinski [13] to calculate the continued fraction development of $\sqrt{D}$, that can be summarized as follows. Let b_i and $c_i > 0, \in \mathbb{Z}$, $x_i \in \mathbb{R}$ irrational numbers, $a_i = \lfloor x_i \rfloor$ for $i \geq 1$, the partial denominators a_i of the continued fraction are then successively

$$a_0 = \left\lfloor \sqrt{D} \right\rfloor \Longrightarrow \sqrt{D} = a_0 + \frac{1}{x_1} = a_0 + \frac{c_1}{\sqrt{D}+b_1}, \ \ b_1 = a_0, \ \ c_1 = D - b_1^2 \tag{2.3}$$

$$a_i = \lfloor x_i \rfloor \Longrightarrow x_i = a_i + \frac{1}{x_{i+1}} = a_i + \left(\frac{\sqrt{D}+b_i}{c_i} - a_i \right) = a_i + \frac{c_{i+1}}{\sqrt{D}+b_{i+1}},$$

$$b_{i+1} = a_i c_i - b_i, \ \ c_{i+1} = \frac{D - b_{i+1}^2}{c_i} \tag{2.4}$$

(see [13], p.313, for more details). Therefore a_{i-1}, b_i and c_i can be calculated successively by

$$a_{i-1} = \left\lfloor \frac{a_0 + b_{i-1}}{c_{i-1}} \right\rfloor, \ \ b_i = a_{i-1}c_{i-1} - b_{i-1}, \ \ c_i = \frac{D - b_i^2}{c_{i-1}} \tag{2.5}$$

with $b_0 = 0$ and $c_0 = 1$, until a_i is found such as $a_i = a_{r+1} = 2a_0$.
The fundamental solution (X_1, Y_1) of the simple Pell equation (1.2) is found for a particular value of D by computing the r^{th}convergent (p_r/q_r) of the continued fraction $[a_0; \overline{a_1, \ldots, a_r, 2a_0}]$ of $\sqrt{D}$. The convergents p_i and q_i can be found by the recurrence relations

$$p_i = a_i p_{i-1} + p_{i-2} \ ; \quad q_i = a_i q_{i-1} + q_{i-2} \tag{2.6}$$

with $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, $q_0 = 1$. The fundamental solution (X_1, Y_1) is then (p_r, q_r) if $r \equiv 1 (mod\, 2)$, or (p_{2r+1}, q_{2r+1}) if $r \equiv 0 (mod\, 2)$, with X_1 and Y_1 forming

an irreducible fraction (X_1/Y_1), where, to recall, $r+1$ is the period length of the continued fraction expansion of $\sqrt{D}$.

3. Theorems on general solutions of the simple Pell equation

A first theorem is demonstrated allowing to find the convergents (p_r/q_r) of $\sqrt{D}$ for $D=(f(u))^2 \pm 2^\alpha n$ where $n>0$ and $\alpha \geq 0, \in \mathbb{Z}$, and $f(u)$ is any polynomial function of $u \in \mathbb{Z}$, taking only positive integer values and such that $f(u) \equiv 0 \left(mod\ \left(2^{\alpha-1}n\right)\right)$ for $\forall u \in \mathbb{Z}$.

Theorem 1. *For squarefree D, n and $r>0, \in \mathbb{Z}$ and $\alpha \geq 0, \in \mathbb{Z}$, let $f(u)$ be any polynomial function of $u \in \mathbb{Z}$ such that $f(u)>0, \in \mathbb{Z}$ for $\forall u \in \mathbb{Z}$ and f is written instead of $f(u)$ for convenience; if*

$$D=f^2 \pm 2^\alpha n \tag{3.1}$$

and

$$f \equiv 0 \left(mod\ \left(2^{\alpha-1}n\right)\right) \tag{3.2}$$

then the convergents (p_r/q_r) of $\sqrt{D}$ are

$$\frac{p_r}{q_r}=\left(\frac{\frac{f^2}{2^{\alpha-1}n}\pm 1}{\frac{f}{2^{\alpha-1}n}}\right) \tag{3.3}$$

and for the case of the minus sign in front of $2^\alpha n$ in (3.1), $f^2>2^\alpha n$ and α and n not taking simultaneously the values $\alpha=0$ and $n=1$.

This theorem is demonstrated by calculating the continued fraction development of $\sqrt{D}$ following Sierpinski's algorithm (2.3) to (2.5) and the convergents (p_r/q_r) of $\sqrt{D}$ by (2.6).

Proof. For squarefree D, n and $r>0, \in \mathbb{Z}$; $\alpha \geq 0, \in \mathbb{Z}$; $f(u)$ any polynomial of $u \in \mathbb{Z}$ taking values $f=f(u)>0, \in \mathbb{Z}$.
(i) For the case of the plus sign in front of $2^\alpha n$ in (3.1), i.e.

$$D=f^2+2^\alpha n \tag{3.4}$$

one has

$$f^2<f^2+2^\alpha n<(f+1)^2 \tag{3.5}$$

under the condition that

$$f \geq 2^{\alpha-1}n \tag{3.6}$$

It yields successively $a_0=\left\lfloor\sqrt{f^2+2^\alpha n}\right\rfloor=f$, $b_1=f$, $c_1=2^\alpha n$; $a_1=\frac{f}{2^{\alpha-1}n}$ with the condition, if $\alpha>0$,

$$f \equiv 0 \left(mod\ \left(2^{\alpha-1}n\right)\right) \tag{3.7}$$

or if $\alpha=0$,

$$2f \equiv 0\,(mod\, n) \tag{3.8}$$

It yields then $b_2=f$, $c_2=1$; $a_2=2f=2a_0$, giving the continued fraction

$$\sqrt{f^2+2^\alpha n}=\left[f;\left(\frac{f}{2^{\alpha-1}n}\right),2f,\ldots\right] \tag{3.9}$$

with $r=1$, which, with (2.6), yields immediately

$$\frac{p_r}{q_r} = \left(\frac{\frac{f^2}{2^{\alpha-1}n}+1}{\frac{f}{2^{\alpha-1}n}}\right) \tag{3.10}$$

(ii) For the case of the minus sign in front of $2^\alpha n$ in (3.1), i.e.

$$D = f^2 - 2^\alpha n \tag{3.11}$$

and α and n not simultaneously $\alpha = 0$ and $n = 1$, one has

$$(f-1)^2 < f^2 - 2^\alpha n < f^2 \tag{3.12}$$

under the condition that

$$f > 2^{\alpha-1}n \tag{3.13}$$

It yields successively $a_0 = \left\lfloor \sqrt{f^2 - 2^\alpha n} \right\rfloor = (f-1)$, $b_1 = (f-1)$, $c_1 = (2f - 2^\alpha n - 1)$; $a_1 = 1$ with the condition

$$f > 2^\alpha n \tag{3.14}$$

One has then $b_2 = (f - 2^\alpha n)$, $c_2 = 2^\alpha n$; $a_2 = \left(\frac{f}{2^{\alpha-1}n} - 2\right)$ with the condition (3.7) if $\alpha > 0$ or (3.8) if $\alpha = 0$. It yields then $b_3 = (f - 2^\alpha n)$, $c_3 = (2f - 2^\alpha n - 1)$; $a_3 = 1$, $b_4 = (f-1)$, $c_4 = 1$; $a_4 = 2\,(f-1) = 2a_0$, giving the continued fraction

$$\sqrt{f^2 - 2^\alpha n} = \left[(f-1)\,; 1, \left(\frac{f}{2^{\alpha-1}n} - 2\right), 1, 2\,(f-1)\,, \ldots\right] \tag{3.15}$$

with $r=3$, which, with (2.6), yields immediately

$$\frac{p_r}{q_r} = \left(\frac{\frac{f^2}{2^{\alpha-1}n}-1}{\frac{f}{2^{\alpha-1}n}}\right) \tag{3.16}$$

□

Note that the conditions (3.2) and $f > 0$ means that $\exists\, k > 0, \in \mathbb{Z}$ such that $f = 2^{\alpha-1}kn$, yielding simpler expressions of D and of the convergents (p_r/q_r) of $\sqrt{D}$ as

$$D = 2^\alpha n \left(2^{\alpha-2}k^2 n \pm 1\right) \tag{3.17}$$

$$\frac{p_r}{q_r} = \left(\frac{2^{\alpha-1}k^2 n \pm 1}{k}\right) \tag{3.18}$$

The following corollary give particular relations deduced from this Theorem, some of them being already given by other authors (see e.g. [13]).

Corollary 2. *For squarefree D, d, m, n and $r > 0, \in \mathbb{Z}$; $\alpha \geq 0, \in \mathbb{Z}$; and for appropriate conditions as in Theorem 1, the following relations hold for the convergents (p_r/q_r) of $\sqrt{D}$:*
(i) for $D = \left(d^2 - 1\right)$,

$$\frac{p_r}{q_r} = \left(\frac{d}{1}\right) \tag{3.19}$$

(ii) for $D = \left(d^2 + 1\right)$,

$$\frac{p_r}{q_r} = \left(\frac{2d^2+1}{2d}\right) \tag{3.20}$$

(iii) for $D = \left(d^2 \pm 2\right)$,

$$\frac{p_r}{q_r} = \left(\frac{d^2 \pm 1}{d}\right) \tag{3.21}$$

(iv) for$D = m^2\left(d^2 \pm 2^\alpha n\right)$,

$$\frac{p_r}{q_r} = \left(\frac{\frac{d^2}{2^{\alpha-1}n} \pm 1}{\frac{d}{2^{\alpha-1}mn}}\right) \tag{3.22}$$

(v) for $D = \left(\left(md^\beta\right)^2 \pm 2^\alpha n\right)$,

$$\frac{p_r}{q_r} = \left(\frac{\frac{\left(md^\beta\right)^2}{2^{\alpha-1}n} \pm 1}{\frac{md^\beta}{2^{\alpha-1}n}}\right) \tag{3.23}$$

(vi) for$D = n\left(nd^2 \pm 1\right)$,

$$\frac{p_r}{q_r} = \left(\frac{2nd^2 \pm 1}{2d}\right) \tag{3.24}$$

(vii) for$D = d\left(m^2 d \pm 2\right)$,

$$\frac{p_r}{q_r} = \left(\frac{m^2 d \pm 1}{m}\right) \tag{3.25}$$

(viii) for $D = d\,(d+4)$, $d \equiv 0\,(mod\,2)$ *and* $d \geq 2$,

$$\frac{p_r}{q_r} = \left(\frac{\frac{(d+2)^2}{2} - 1}{\frac{d+2}{2}}\right) \tag{3.26}$$

(ix) for $D = \left((d \pm 2)^2 \pm 4\right)$, $d \equiv 0\,(mod\,2)$ *and* $d \geq 2$,

$$\frac{p_r}{q_r} = \left(\frac{\frac{(d\pm 2)^2}{2} \pm 1}{\frac{d\pm 2}{2}}\right) \tag{3.27}$$

Proof. For squarefree D, d, d', m, n, n' and $r > 0, \in \mathbb{Z}$; t, α and $\beta \geq 0, \in \mathbb{Z}$; let

$$f\,(d) = \left(md^\beta \pm t\right) \tag{3.28}$$

be a polynomial of d such that $f\,(d) > 0, \in \mathbb{Z}$. One has then the following from Theorem 1, yielding the respective convergents by (2.6):
(i) let $t = \alpha = 0$, and $m = n = \beta = 1$ in (3.28) and (3.1), yielding $f\,(d) = d$ and $D = d^2 - 1$. Then, in the proof (ii) of Theorem 1, $a_2 = 2\,(d-1)) = 2a_0$ for $d \geq 2$, and the continued fraction reduces to $\sqrt{d^2 - 1} = [(d-1)\,; 1, 2\,(d-1)\,, \ldots]$, with $r = 1$, yielding directly (3.19) by $(2.6)^2$. Note that (3.3) yields a following convergent for $r' = 2r + 1 = 3$, which provides a next solution to the Pell equation.
(ii) let $t = \alpha = 0$, and $m = n = \beta = 1$ in (3.28) and (3.1), yielding $f\,(d) = d$ and $D = d^2 + 1$; then (3.3) yields immediately (3.20);
(iii) let $t = 0$, and $m = n = \alpha = \beta = 1$ in (3.28) and (3.1) yielding $f\,(d) = d$ and $D = d^2 \pm 2$, giving immediately (3.21) from (3.3), with, for the minus sign,

[2] In [13] p.321, it is erroneously stated that "$\sqrt{a^2+1} = [a; a, 2a, \ldots]$ (instead of $[a; 2a, \ldots]$) for any natural number a" and "$\sqrt{(na^2)^2+1}$ (instead of $\sqrt{(na)^2+1}$) $= [(na-1)\,; 1, 2n-2, 1, 2\,(na-1)\,, \ldots]$ for natural numbers a and $n > 1$"

the condition that $d \geq 3$; if $d = 2$, the continued fraction (3.15) reduces to $\sqrt{2} = [1; 2, \ldots]$ with $r = 1$, yielding $(p_1/q_1) = (3/2)$ by (2.6).
(iv) let $t = 0$, $\beta = 1$ and $n = n'm^2$ in (3.28) and (3.1), yielding $f(d) = md$ and $D = m^2\left(d^2 \pm 2^\alpha n'\right)$ and dropping the $'$; then (3.22) is immediate from (3.3) with the condition that $d \equiv 0\left(mod\ \left(2^{\alpha-1}nm\right)\right)$;
(v) let $t = 0$ in (3.28), yielding $f(d) = md^\beta$; then (3.23) is immediate from (3.3), with the condition that $md^\beta \equiv 0\left(mod\ \left(2^{\alpha-1}n\right)\right)$;
(vi) setting $t = \alpha = 0$, $\beta = 1$ and $m = n$ in (3.23) yields immediately (3.24);
(vii) let $t = 0$ and $\beta = 1$ and either $\alpha = 1$ and $n = d$, or $\alpha = 0$ and $n = 2d$, then (3.25) is immediate from (3.3);
(viii) and (ix) let m, n and $\beta = 1$, t and $\alpha = 2$, and $d \equiv 0\,(mod\,2)$ with $d \geq 2$; then (3.26) and (3.27) are immediate from (3.3). □

Another theorem is demonstrated allowing to calculate the convergents (p_r/q_r) of $\sqrt{D}$ for $D = (f(u))^2 \pm 2^\alpha n$ where $f(u)$ is any polynomial function of u, taking only positive integer values and such that $f(u) \equiv \left(2^{\alpha-2}n\right)\left(mod\ \left(2^{\alpha-1}n\right)\right)$ for $\forall u \in \mathbb{Z}$, with $n > 0$ and $\alpha \geq 0, \in \mathbb{Z}$.

Theorem 3. *For squarefree D, n and $r > 0, \in \mathbb{Z}$ and $\alpha \geq 0, \in \mathbb{Z}$, let $f(u)$ be any polynomial function of $u \in \mathbb{Z}$ such that $f(u) > 0, \in \mathbb{Z}$ for $\forall u \in \mathbb{Z}$ and f is written instead of $f(u)$ for convenience; if*

$$D = f^2 \pm 2^\alpha n \tag{3.29}$$

and

$$f \equiv \left(2^{\alpha-2}n\right)\left(mod\ \left(2^{\alpha-1}n\right)\right) \tag{3.30}$$

then the convergents (p_r/q_r) of $\sqrt{D}$ are, if $f \equiv 0\,(mod\,2)$,

$$\frac{p_r}{q_r} = \left(\frac{\frac{f^2\left(f^2 \pm 2^\alpha n\right)}{2^{2\alpha-3}n^2} + 1}{\frac{f\left(f^2 \pm 2^{\alpha-1}n\right)}{2^{2\alpha-3}n^2}}\right) \tag{3.31}$$

and, if $f \equiv 1\,(mod\,2)$,

$$\frac{p_r}{q_r} = \left(\frac{\frac{f^2\left(f^2 \pm 3\times 2^{\alpha-2}n\right)^2}{2^{3\alpha-5}n^3} \pm 1}{\frac{f\left(f^2 \pm 2^{\alpha-2}n\right)\left(f^2 \pm 3\times 2^{\alpha-2}n\right)}{2^{3\alpha-5}n^3}}\right) \tag{3.32}$$

under the conditions for $D = \left(f^2 + 2^\alpha n\right)$ and $f \equiv 0$ or $1\,(mod\,2)$,

$$\text{if} \quad 1 \leq n \leq 2^{2-\alpha} \quad \text{then } f \geq 2^{\alpha-1}n \tag{3.33}$$

$$\text{if} \quad n > 2^{2-\alpha} \quad \text{then } f \geq \left(3 \times 2^{\alpha-2}n - 1\right) \tag{3.34}$$

for $D = \left(f^2 - 2^\alpha n\right)$ and $f \equiv 0\,(mod\,2)$,

$$\text{if} \quad \alpha \geq 2, \text{ or if } \alpha = 0 \text{ or } 1 \text{ and } 3n \geq 2^{3-\alpha} \quad \text{then } f \geq 2^\alpha n \tag{3.35}$$

$$\text{if} \quad \alpha = 0 \text{ or } 1 \text{ and } 3n < 2^{3-\alpha} \quad \text{then } f \geq \left(2^{\alpha-2}n + 2\right) \tag{3.36}$$

for $D = \left(f^2 - 2^\alpha n\right)$ and $f \equiv 1\,(mod\,2)$,

$$\text{if} \quad n \geq \left(2^{2-\alpha} + 1\right) \quad \text{then } f \geq 3 \times 2^{\alpha-2}n \tag{3.37}$$

$$\text{if} \quad 1 \leq n \leq 2^{2-\alpha} \quad \text{then } f > 2^{\alpha-3}n + 3 \tag{3.38}$$

Proof. For squarefree D, n and $r>0,\in\mathbb{Z}$; $\alpha\geq 0,\in\mathbb{Z}$; $f\left(u\right)$ any polynomial of $u\in\mathbb{Z}$ taking values $f=f\left(u\right)>0,\in\mathbb{Z}$, let

$$f\equiv\left(2^{\alpha-2}n\right)\left(mod\ \left(2^{\alpha-1}n\right)\right)\tag{3.39}$$

Obviously for $\alpha\geq 3$, $f\equiv 0\,(mod\,2)$ always, while for $0\leq\alpha\leq 2$, $f>0,\in\mathbb{Z}$ if $n\equiv 0\left(mod\,2^{2-\alpha}\right)$ and $f\equiv 0\,(mod\,2)$ if $n\equiv 0\left(mod\,2^{3-\alpha}\right)$, or $f\equiv 1\,(mod\,2)$ if $n\equiv 2^{2-\alpha}\left(mod\,2^{3-\alpha}\right)$.

(i) For the case of the plus sign in front of $2^{\alpha}n$ in (3.29), i.e.

$$D=\left(f^{2}+2^{\alpha}n\right)\tag{3.40}$$

one has

$$f^{2}<f^{2}+2^{\alpha}n<\left(f+1\right)^{2}\tag{3.41}$$

under the condition that

$$f\geq 2^{\alpha-2}n\tag{3.42}$$

It yields successively $a_0=\left\lfloor\sqrt{f^{2}+2^{\alpha}n}\right\rfloor=f$, $b_1=f$, $c_1=2^{\alpha}n$; $a_1=\left(\frac{f-2^{\alpha-2}n}{2^{\alpha-1}n}\right)$, $b_2=\left(f-2^{\alpha-1}n\right)$, $c_2=\left(f-2^{\alpha-2}n+1\right)$; $a_2=1$ with the condition that

$$f>2^{\alpha-2}n+1\tag{3.43}$$

Then $b_3=(2^{\alpha-2}n+1)$, $c_3=(f+2^{\alpha-2}n-1)$; $a_3=1$, $b_4=(f-2)$, $c_4=4$; $a_4=\left\lfloor\frac{f-1}{2}\right\rfloor$.

(i.1) If $f\equiv 0\,(mod\,2)$, $a_4=\left(\frac{f-2}{2}\right)$, $b_5=(f-2)$, $c_5=\left(f+2^{\alpha-2}n-1\right)$; $a_5=1$ with the condition (3.43). Then $b_6=\left(2^{\alpha-2}n+1\right)$, $c_6=\left(f-2^{\alpha-2}n+1\right)$; $a_6=1$ with the condition that

$$f>3\times 2^{\alpha-2}n-1\tag{3.44}$$

and $b_7=\left(f-2^{\alpha-1}n\right)$, $c_7=2^{\alpha}n$; $a_7=\left(\frac{f-2^{\alpha-2}n}{2^{\alpha-1}n}\right)$, $b_8=f$, $c_8=1$; $a_8=2f=2a_0$, yielding the continued fraction

(3.45)
$$\sqrt{f^{2}+2^{\alpha}n}=\left[f;\left(\frac{f-2^{\alpha-2}n}{2^{\alpha-1}n}\right),1,1,\left(\frac{f-2}{2}\right),1,1,\left(\frac{f-2^{\alpha-2}n}{2^{\alpha-1}n}\right),2f,\ldots\right]$$

with $r=7$, which by (2.6) yields

$$\frac{p_7}{q_7}=\left(\frac{\frac{f^{2}\left(f^{2}+2^{\alpha}n\right)}{2^{2\alpha-3}n^{2}}+1}{\frac{f\left(f^{2}+2^{\alpha-1}n\right)}{2^{2\alpha-3}n^{2}}}\right)\tag{3.46}$$

(i.2) If $f\equiv 1\,(mod\,2)$, $a_4=\left(\frac{f-1}{2}\right)$. Then, $b_5=f$, $c_5=2^{\alpha-2}n$; $a_5=\left(\frac{f}{2^{\alpha-3}n}\right)$, $b_6=f$, $c_6=4$; $a_6=\left(\frac{f-1}{2}\right)$; $b_7=(f-2)$, $c_7=\left(f+2^{\alpha-2}n-1\right)$; $a_7=1$ with the condition (3.43), $b_8=\left(2^{\alpha-2}n+1\right)$, $c_8=\left(f-2^{\alpha-2}n+1\right)$; $a_8=1$ with the condition (3.44), $b_9=\left(f-2^{\alpha-1}n\right)$, $c_9=2^{\alpha}n$; $a_9=\left(\frac{f-2^{\alpha-2}n}{2^{\alpha-1}n}\right)$, $b_{10}=f$, $c_{10}=1$;

$a_{10} = 2f = 2a_0$, yielding the continued fraction

$$\begin{aligned} \sqrt{f^2+2^\alpha n} &= \left[f; \left(\frac{f-2^{\alpha-2}n}{2^{\alpha-1}n}\right), 1, 1, \left(\frac{f-1}{2}\right), \left(\frac{f}{2^{\alpha-3}n}\right), \right. \\ &\left. \left(\frac{f-1}{2}\right), 1, 1, \left(\frac{f-2^{\alpha-2}n}{2^{\alpha-1}n}\right), 2f, \dots\right] \end{aligned} \tag{3.47}$$

with $r = 9$, which by (2.6) yields

$$\frac{p_9}{q_9} = \left(\frac{\frac{f^2\left(f^2+3\times 2^{\alpha-2}n\right)^2}{2^{3\alpha-5}n^3}+1}{\frac{f\left(f^2+2^{\alpha-2}n\right)\left(f^2+3\times 2^{\alpha-2}n\right)}{2^{3\alpha-5}n^3}}\right) \tag{3.48}$$

Summarizing the conditions (3.42) to (3.44) for $f \equiv 0\,(mod\,2)$ or $f \equiv 1\,(mod\,2)$, one has that
- if $1 \leq n \leq 2^{2-\alpha}$, then

$$f \geq 2^{\alpha-1}n \tag{3.49}$$

- if $n > 2^{2-\alpha}$, then

$$f > 3\times 2^{\alpha-2}n - 1 \tag{3.50}$$

(ii) For the case of the minus sign in front of $2^\alpha n$ in (3.29), i.e.

$$D = \left(f^2 - 2^\alpha n\right) \tag{3.51}$$

one has

$$(f-1)^2 < f^2 - 2^\alpha n < f^2 \tag{3.52}$$

giving the same condition as (3.42). It yields successively $a_0 = \left\lfloor\sqrt{f^2-2^\alpha n}\right\rfloor = (f-1)$, $b_1 = (f-1)$, $c_1 = (2f - 2^\alpha n - 1)$; $a_1 = 1$ with the condition that

$$f > 2^\alpha n \tag{3.53}$$

Then, $b_2 = (f - 2^\alpha n)$, $c_2 = 2^\alpha n$; $a_2 = \left(\frac{f-3\times 2^{\alpha-2}n}{2^{\alpha-1}n}\right)$, $b_3 = \left(f - 2^{\alpha-1}n\right)$, $c_3 = \left(f - 2^{\alpha-2}n - 1\right)$; $a_3 = 2$ with the condition that

$$f > 2^{\alpha-2}n + 2 \tag{3.54}$$

and $b_4 = (f-2)$, $c_4 = 4$; $a_4 = \left\lfloor\frac{2f-3}{4}\right\rfloor$.

(ii.1) If $f \equiv 0\,(mod\,2)$, $a_4 = \left(\frac{f-2}{2}\right)$. Then, $b_5 = (f-2)$, $c_5 = \left(f - 2^{\alpha-2}n - 1\right)$; $a_5 = 2$ with the condition that

$$f > 3\times 2^{\alpha-2}n \tag{3.55}$$

Then, $b_6 = \left(f - 2^{\alpha-1}n\right)$, $c_6 = 2^\alpha n$; $a_6 = \left(\frac{f-3\times 2^{\alpha-2}n}{2^{\alpha-1}n}\right)$, $b_7 = (f - 2^\alpha n)$, $c_7 = (2f - 2^\alpha n - 1)$; $a_7 = 1$, $b_8 = (f-1)$, $c_8 = 1$; $a_8 = 2\,(f-1) = 2a_0$, yielding the continued fraction

$$\begin{aligned} \sqrt{f^2-2^\alpha n} &= \left[(f-1); 1, \left(\frac{f-3\times 2^{\alpha-2}n}{2^{\alpha-1}n}\right), 2, \left(\frac{f-2}{2}\right), \right. \\ &\left. 2, \left(\frac{f-3\times 2^{\alpha-2}n}{2^{\alpha-1}n}\right), 1, 2\,(f-1), \dots\right] \end{aligned} \tag{3.56}$$

with $r = 7$, which by (2.6) yields

$$\frac{p_7}{q_7} = \left(\frac{\frac{f^2\left(f^2-2^{\alpha}n\right)}{2^{2\alpha-3}n^2} + 1}{\frac{f\left(f^2-2^{\alpha-1}n\right)}{2^{2\alpha-3}n^2}} \right) \tag{3.57}$$

Summarizing the conditions (3.42) and (3.53) to (3.55) for $f \equiv 0\,(mod\,2)$, one has that
- if $\alpha \geq 2$, or if $\alpha = 0$ or 1 and $3n \geq 2^{3-\alpha}$, then

$$f \geq 2^{\alpha}n \tag{3.58}$$

- if $\alpha = 0$ or 1 and $3n < 2^{3-\alpha}$, then

$$f > 2^{\alpha-2}n + 2 \tag{3.59}$$

(ii.2) If $f \equiv 1\,(mod\,2)$, $a_4 = \left(\frac{f-3}{2}\right)$, $b_5 = (f-4)$, $c_5 = \left(2f - 2^{\alpha-2}n - 4\right)$; $a_5 = 1$ with the condition that

$$f \geq 2^{\alpha-1}n + 1 \tag{3.60}$$

and $b_6 = \left(f - 2^{\alpha-2}n\right)$, $c_6 = 2^{\alpha-2}n$; $a_6 = \left(\frac{f-2^{\alpha-2}n}{2^{\alpha-3}n}\right)$; $b_7 = \left(f - 2^{\alpha-2}n\right)$, $c_7 = \left(2f + 2^{\alpha-2}n - 4\right)$; $a_7 = 1$ with the condition that

$$f \geq 2^{\alpha-3}n + 3 \tag{3.61}$$

Then $b_8 = (f-4)$, $c_8 = 4$; $a_8 = \left(\frac{f-3}{2}\right)$, $b_9 = (f-2)$, $c_9 = \left(f - 2^{\alpha-2}n - 1\right)$; $a_9 = 2$ with the condition (3.55), $b_{10} = \left(f - 2^{\alpha-1}n\right)$, $c_{10} = 2^{\alpha}n$; $a_{10} = \left(\frac{f-3\times2^{\alpha-2}n}{2^{\alpha-1}n}\right)$, $b_{11} = (f - 2^{\alpha}n)$, $c_{11} = (2f - 2^{\alpha}n - 1)$; $a_{11} = 1$, $b_{12} = (f-1)$, $c_{12} = 1$; $a_{12} = 2\,(f-1) = 2a_0$, yielding the continued fraction

$$\begin{aligned} \sqrt{f^2 - 2^{\alpha}n} \quad = \quad & \Bigg[(f-1)\,; 1, \left(\frac{f - 3\times2^{\alpha-2}n}{2^{\alpha-1}n}\right), 2, \left(\frac{f-3}{2}\right), 1, \\ & \left(\frac{f - 2^{\alpha-2}n}{2^{\alpha-3}n}\right), 1, \left(\frac{f-3}{2}\right), 2, \\ & \left(\frac{f - 3\times2^{\alpha-2}n}{2^{\alpha-1}n}\right), 1, 2\,(f-1), \ldots\Bigg] \end{aligned} \tag{3.62}$$

with $r = 11$, which by (2.6) yields

$$\frac{p_{11}}{q_{11}} = \left(\frac{\frac{f^2\left(f^2-3\times2^{\alpha-2}n\right)^2}{2^{3\alpha-5}n^3} - 1}{\frac{f\left(f^2-2^{\alpha-2}n\right)\left(f^2-3\times2^{\alpha-2}n\right)}{2^{3\alpha-5}n^3}} \right) \tag{3.63}$$

Summarizing the conditions (3.42), (3.60), (3.61) and (3.55) for $f \equiv 1\,(mod\,2)$ and as $0 \leq \alpha \leq 2$, one has that
- if $n \geq 2^{2-\alpha} + 1$, then

$$f \geq 3 \times 2^{\alpha-2}n \tag{3.64}$$

- if $1 \leq n \leq 2^{2-\alpha}$, then

$$f > 2^{\alpha-3}n + 3 \tag{3.65}$$

$\square$

Like above, note that the conditions (3.30) and $f > 0$ means that $\exists\, k > 0, \in \mathbb{Z}$ such that $f = 2^{\alpha-2}(2k+1)\, n$, yielding from (3.29)

$$D = 2^{\alpha} n \left(2^{\alpha-4} (2k+1)^2 n \pm 1 \right) \tag{3.66}$$

and for the plus sign in front of 1 in (3.66), conditions (3.33) and (3.34) reduce to $k \geq 1$, and for the minus sign in front of 1 in (3.66), for $f \equiv 0\,(mod\,2)$, (3.35) and (3.36) reduce respectively to $k \geq 0$ and $k \geq 2$, and for $f \equiv 1\,(mod\,2)$, (3.37) and (3.38) reduce respectively to $k \geq 1$ and $k \geq 2$. Expressions simpler than (3.31) and (3.32) for the convergents (p_r/q_r) of $\sqrt{D}$ are then

$$\frac{p_r}{q_r} = \left(\frac{2^{\alpha-1}(2k+1)^2 n \left(2^{\alpha-4}(2k+1)^2 n \pm 1 \right) + 1}{(2k+1)\left(2^{\alpha-3}(2k+1)^2 n \pm 1 \right)} \right) \tag{3.67}$$

if $f = 2^{\alpha-2}(2k+1)\, n \equiv 0\,(mod\,2)$ (i.e. for either $\alpha \geq 3$, or if $\alpha = 0$, $n \equiv 0\,(mod\,8)$, or if $\alpha = 1$, $n \equiv 0\,(mod\,4)$, or if $\alpha = 2$, $n \equiv 0\,(mod\,2)$), and

$$\frac{p_r}{q_r} = \left(\frac{2^{\alpha-3}(2k+1)^2 n \left(2^{\alpha-2}(2k+1)^2 n \pm 3 \right) \pm 1}{\frac{1}{2}(2k+1)\left(2^{\alpha-2}(2k+1)^2 n \pm 1 \right)\left(2^{\alpha-2}(2k+1)^2 n \pm 3 \right)} \right) \tag{3.68}$$

if $f = 2^{\alpha-2}(2k+1)\, n \equiv 1\,(mod\,2)$ (i.e. for $\alpha \leq 2$, if $\alpha = 0$, $n \equiv 4\,(mod\,8)$, or if $\alpha = 1$, $n \equiv 2\,(mod\,4)$, or if $\alpha = 2$, $n \equiv 1\,(mod\,2)$).
Several particular relations can also be deduced from the general cases given in this Theorem.

Corollary 4. *For squarefree D, d, m, n and $r > 0, \in \mathbb{Z}$, $d \equiv 1\,(mod\,2)$; $\alpha \geq 0, \in \mathbb{Z}$; and for appropriate conditions as in Theorem 3, the following relations hold for the convergents (p_r/q_r) of $\sqrt{D}$:*
(i) for $D = 4d\,(d \pm 2)$ and, in the case of the minus sign, $d \geq 3$,

$$\frac{p_r}{q_r} = \left(\frac{2d\,(d \pm 2) + 1}{d \pm 1} \right) \tag{3.69}$$

(ii) for $D = 4\left(d^2 \pm 2\right)$ and, in the case of the minus sign, $d \geq 3$,

$$\frac{p_r}{q_r} = \left(\frac{2d^2\left(d^2 \pm 2\right) + 1}{d\,(d^2 \pm 1)} \right) \tag{3.70}$$

(iii) for $D = 16n\left(nd^2 \pm 1\right)$ and, in the case of the minus sign, $nd > 1$,

$$\frac{p_r}{q_r} = \left(\frac{8nd^2\left(nd^2 \pm 1\right) + 1}{d\,(2nd^2 \pm 1)} \right) \tag{3.71}$$

(iv) for $D = 4n\left(nd^2 \pm 2\right)$ and, in the case of the minus sign, $nd > 2$,

$$\frac{p_r}{q_r} = \left(\frac{2nd^2\left(nd^2 \pm 2\right) + 1}{d\,(nd^2 \pm 1)} \right) \tag{3.72}$$

(v) for $D = \left(d^2 \pm 4\right)$ and, in the case of the minus sign, $d \geq 3$

$$\frac{p_r}{q_r} = \left(\frac{\frac{d^2\left(d^2 \pm 3\right)^2}{2} \pm 1}{\frac{d(d^2 \pm 1)(d^2 \pm 3)}{2}} \right) \tag{3.73}$$

(vi) for $D = n\left(nd^2 \pm 4\right)$, $n \equiv 1\,(mod\,2)$ *and, in the case of the minus sign,* $nd^2 > 2$,

$$\frac{p_r}{q_r} = \left(\frac{\frac{nd^2\left(nd^2 \pm 3\right)^2}{2} \pm 1}{\frac{d\left(nd^2 \pm 1\right)\left(nd^2 \pm 3\right)}{2}}\right) \tag{3.74}$$

Proof. For squarefree D, d, m, n, n' and $r > 0, \in \mathbb{Z}$, $d \equiv 1\,(mod\,2)$, α and $\beta \geq 0, \in \mathbb{Z}$; let

$$f\,(d) = md^{\beta} \tag{3.75}$$

such that $f\,(d) > 0, \in \mathbb{Z}$. One has then the following from Theorem 3, yielding the respective convergents by (2.6):
(i) immediate for $m = 2$, $\beta = 1$, $\alpha = 3$ and $n = d$ in (3.75) and (3.31).
(ii) immediate for $m = 2$, $\beta = 1$,$\alpha = 3$ and $n = 1$ in (3.75) and (3.31).
(iii) immediate for $m = 2n$, $\beta = 1$,$\alpha = 3$ and $n = 2n'$ and dropping the prime sign $'$, in (3.75) and (3.31).
(iv) immediate for $m = 2n$, $\beta = 1$ and $\alpha = 3$ in (3.75) and (3.31).
(v) immediate for $m = 1$, $\beta = 1$, $\alpha = 2$ and $n = 1$ in (3.75) and (3.32).
(vi) immediate for $m = n$, $\beta = 1$ and $\alpha = 2$ in (3.75) and (3.32). □

For the cases (i) to (iv), it is easy to see that D is always $4\,(mod\,8)$. From case (iv), generalizing cases (i) to (iii), $D = 4n\left(nd^2 \pm 2\right) = \left((2nd)^2 \pm 8n\right)$ and it is easy to see that, as $d \equiv 1\,(mod\,2)$, if $n \equiv 0$ or $1\,(mod\,2)$ then $(2nd) \equiv 0$ or $2\,(mod\,4)$, i.e. solutions will be found for even integer squares $(2nd)^2$ different from powers of 2 ($(2nd)^2 \neq 2^{\alpha}$), plus or minus even or odd multiples of 8 depending on the 0 or $2\,(mod\,4)$ value of $2nd$. For example, for $n \equiv 1\,(mod\,2)$, $D = 6^2 \pm 8$, $10^2 \pm 8$, $14^2 \pm 8$, ... and also $D = 30^2 \pm 8$, $30^2 \pm 24$, $30^2 \pm 40$, ...; for $n \equiv 0\,(mod\,2)$, $D = 12^2 \pm 16$, $20^2 \pm 16$, $24^2 \pm 32$, $28^2 \pm 16$, ...
The case (v) of $D = \left(d^2 + 4\right)$ with $d \equiv 1\,(mod\,2)$ is interesting as it gives the solutions for all the values of D of the form $D = (8\triangle\,(k) + 5) = \left((2k+1)^2 + 4\right)$ with $k \geq 0, \in \mathbb{Z}$, i.e. $D \equiv 5\,(mod\,8)$ and $\frac{(D-5)}{8} = \triangle\,(k)$, where $\triangle\,(k)$ is the triangular number of k. Similarly, the case (v) of $D = \left(d^2 - 4\right)$ with $d \equiv 1\,(mod\,2)$ gives the solutions for all the values of D of the form $D = (8\triangle\,(k) - 3) = \left((2k+1)^2 - 4\right)$ with $k \geq 1, \in \mathbb{Z}$, i.e. $D \equiv -3\,(mod\,8)$ and $\frac{(D+3)}{8} = \triangle\,(k)$. For example, $D = 3^2 + 4$, $5^2 \pm 4$, $7^2 \pm 4$, $9^2 \pm 4$, ...
The case (vi) is also interesting as it generalizes the previous case (v). D can indistinctly be written as $D = n\left(nd^2 \pm 4\right) = \left(8\triangle\left(\frac{nd-1}{2}\right) + 1 \pm 4n\right)$, which is always $D \equiv 5\,(mod\,8)$ as d and $n \equiv 1\,(mod\,2)$. For example, $D = 9^2 \pm 12$, $15^2 \pm 12$, $15^2 \pm 20$, $21^2 \pm 12$, $21^2 \pm 28$, ...

4. Conclusions

The case considered in these two theorems of any polynomial function $f\,(u)$ of $u \in \mathbb{Z}$ can easily be extended to any function $g\,(x)$ of $x \in \mathbb{R}$ or $h\,(z)$ of $z \in \mathbb{C}$ as long as $g\,(x)$ or $h\,(z) > 0, \in \mathbb{Z}$ for $\forall x \in \mathbb{R}$ or $\forall z \in \mathbb{C}$.
These two general theorems and the various relations in the two corollaries allow the immediate calculation of the fundamental solutions of the simple Pell equation for most of the values of D. A quick survey shows that these two theorems provide

solutions for all squarefree integer values of D for $D \leq 10$, for 72.2% of the values of D for $D < 10^2$, for 35.4% for $D \leq 10^3$, and for 15.7% for $D < 10^4$.

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